Schematic Harder-Narasimhan Stratification

Nitin Nitsure

Abstract

For any flat family of pure-dimensional coherent sheaves on a family of projective schemes, the Harder-Narasimhan type (in the sense of Gieseker semistability) of its restriction to each fiber is known to vary semicontinuously on the parameter scheme of the family. This defines a stratification of the parameter scheme by locally closed subsets, known as the Harder-Narasimhan stratification.

In this note, we show how to endow each Harder-Narasimhan stratum with the structure of a locally closed subscheme of the parameter scheme, which enjoys the universal property that under any base change the pullback family admits a relative Harder-Narasimhan filtration with a given Harder-Narasimhan type if and only if the base change factors through the schematic stratum corresponding to that Harder-Narasimhan type.

The above schematic stratification induces a stacky stratification on the algebraic stack of pure-dimensional coherent sheaves. We deduce that coherent sheaves of a fixed Harder-Narasimhan type form an algebraic stack in the sense of Artin.

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1. Introduction

Let X be a projective scheme over a locally noetherian base scheme S, with a chosen relatively ample line bundle $\mathcal{O}_X(1)$. Let E be a coherent sheaf on X which is flat over S, such that the restriction $E_s = E|_{X_s}$ of E to the schematic fiber X_s of X over each $s \in S$ is a pure-dimensional sheaf of a fixed dimension $d \geq 0$. For any $s \in S$, let $\mathrm{HN}(E_s)$ denote the Harder-Narasimhan type of E_s in the sense of Gieseker semistability. With respect to a certain natural partial order on the set HNT of all possible Harder-Narasimhan types τ, the Harder-Narasimhan function $s \mapsto \mathrm{HN}(E_s)$ is known to be upper semicontinuous on S.

In this note, we prove that each level set $S^{\tau}(E)$ of the Harder-Narasimhan function has a natural structure of a locally closed subscheme of S, with the following universal property: a morphism $T \to S$ factors via $S^{\tau}(E)$ if and only if the pullback E_t on X_t for each $t \in T$ is of type τ and the pullback family E_T on $X \times_S T$ admits a relative Harder-Narasimhan filtration, that is, a filtration $0 \subset F_1 \subset \ldots \subset F_{\ell} = E_T$ by coherent subsheaves such that the graded pieces F_i/F_{i-1} are flat over T, which for each $t \in T$ restricts to the Harder-Narasimhan filtration of E_t on X_t.

As a corollary, we deduce that sheaves of a fixed Harder-Narasimhan type form an algebraic stack in the sense of Artin.

In Section 2 we recall the basic definitions and results of Harder-Narasimhan-Shatz that we need. In Section 3 we prove our main result (Theorem 5), which gives natural schematic structures on the Harder-Narasimhan strata. In Section 4, we show (Theorem 8) that sheaves of a given Harder-Narasimhan type form an Artin algebraic stack.

This work had its origin in questions arising from the proposal of Leticia Brambila-Paz to construct a moduli scheme for indecomposable unstable rank 2 vector bundles on a curve, fixing their Harder-Narasimhan type and the dimension of their vector space global endomorphisms. A construction of such a moduli scheme is given in [B-M-Ni], which uses special cases of the results proved here.

2. The Harder-Narasimhan filtration and stratification

Let $\mathbb{Q}[\lambda]$ be the polynomial ring in the variable λ. An element $f \in \mathbb{Q}[\lambda]$ is called a **numerical polynomial** if $f(\mathbb{Z}) \subset \mathbb{Z}$. If a nonzero numerical polynomial f has degree d, it can be uniquely expanded as $f = (r(f)/d!)\lambda^d +$ lower degree terms, where $r(f) \in \mathbb{Z}$. If $f = 0$ we put $r(f) = 0$. There is a **total order** $\leq$ on $\mathbb{Q}[\lambda]$ under which $f \leq g$ if $f(m) \leq g(m)$ for all sufficiently large integers m. Let the **set of all Harder-Narasimhan types**, denoted by HNT, be the set consisting of all finite sequences $(f_1, \ldots, f_p)$ of numerical polynomials in $\mathbb{Q}[\lambda]$, where p is allowed to vary over all integers ≥ 1, such that the following three conditions are satisfied.

(1) We have $0 < f_1 < \ldots < f_p$ in $\mathbb{Q}[\lambda]$,

(2) the polynomials f_i are all of the same degree, say d, and

(3) the following inequalities are satisfied

$$\frac{f_1}{r(f_1)} > \frac{f_2 - f_1}{r(f_2) - r(f_1)} > \ldots > \frac{f_p - f_{p-1}}{r(f_p) - r(f_{p-1})}.$$

Given any $x = (a, f)$ and $y = (b, g)$ in $\mathbb{Z} \times \mathbb{Q}[\lambda]$, the **segment joining** x **and** y is the subset $\overline{xy} \subset \mathbb{Z} \times \mathbb{Q}[\lambda]$, consisting of all (c, h) such that $(c, h) = t(a, f) + (1-t)(b, g)$ for some $t \in \mathbb{Q}$ with $0 \leq t \leq 1$. For any $(f_1, \ldots, f_p)$ in HNT, we define the corresponding **Harder-Narasimhan polygon** to be the subset

$$\mathrm{HNP}(f_1, \ldots, f_p) \subset \mathbb{Z} \times \mathbb{Q}[\lambda]$$

which is the union of the segments $\overline{x_0 x_1} \cup \overline{x_1 x_2} \cup \ldots \cup \overline{x_{p-1} x_p}$ where $x_0 = (0, 0)$ and $x_i = (r(f_i), f_i)$ for $1 \leq i \leq p$.

A point $(a, f) \in \mathbb{Z} \times \mathbb{Q}[\lambda]$ is said to **lie under** another point $(b, g) \in \mathbb{Z} \times \mathbb{Q}[\lambda]$ if $a = b$ in $\mathbb{Z}$ and $f \leq g$ in $\mathbb{Q}[\lambda]$. A point $(a, f) \in \mathbb{Z} \times \mathbb{Q}[\lambda]$ is said to **lie under the**

polygon $\mathrm{HNP}(g_1, \ldots, g_q)$ if there exists some $(b, g) \in \mathrm{HNP}(g_1, \ldots, g_q)$ such that the point (a, f) lies under the point (b, g). There is a **partial order** $\leq$ on HNT, under which $(f_1, \ldots, f_p) \leq (g_1, \ldots, g_q)$ if for each $1 \leq i \leq p$, the point $(r(f_i), f_i)$ lies under $\mathrm{HNP}(g_1, \ldots, g_q)$.

With the above numerical preliminaries, we now briefly recall the theory of Harder-Narasimhan-Shatz for filtrations and stratifications. For an exposition the reader can see, for example, the book of Huybrechts and Lehn [Hu-Le].

Let Y be a projective scheme over a field k, together with an ample line bundle $\mathcal{O}_Y(1)$. For any coherent sheaf E on Y, we denote by $P(E) \in \mathbb{Q}[\lambda]$ the resulting Hilbert polynomial of E, defined by $P(E)(m) = \sum_i (-1)^i \dim_k H^i(Y, E(m))$. This is a numerical polynomial, and the integer $r(P(E))$ is denoted simply by $r(E)$ (this is called the **rank** of E). For a nonzero E, the degree d of $P(E)$ equals the dimension of the support of E. A coherent sheaf E on Y is said to be **pure-dimensional** of dimension d if for every open subscheme $U \subset Y$ and nonzero coherent subsheaf $F \subset E|_U$, the support of F is of the same dimension d. A pure-dimensional coherent sheaf E is called **semistable** (in the sense of Gieseker) w.r.t. $\mathcal{O}_Y(1)$ if for all coherent subsheaves $F \subset E$, we have $r(E)P(F) \leq r(F)P(E)$.

When E is of pure dimension $d \geq 0$ but not necessarily semistable, it admits a unique strictly increasing filtration $0 = \mathrm{HN}_0(E) \subset \mathrm{HN}_1(E) \subset \ldots \subset \mathrm{HN}_\ell(E) = E$ by coherent subsheaves $\mathrm{HN}_i(E)$ such that for each $1 \leq i \leq \ell$, the graded piece $\mathrm{Gr}_i(E) = \mathrm{HN}_i(E)/\mathrm{HN}_{i-1}(E)$ is semistable of pure dimension d, and the inequalities

$$\frac{P(\mathrm{Gr}_1(E))}{r(\mathrm{Gr}_1(E))} > \ldots > \frac{P(\mathrm{Gr}_\ell(E))}{r(\mathrm{Gr}_\ell(E))}$$

hold. This filtration is called the **Harder-Narasimhan filtration** of E (in the sense of Gieseker semistability). The first step $\mathrm{HN}_1(E)$ is called the **maximal destabilizing subsheaf** of E. The integer ℓ (also written as $\ell(E)$) is called as the **length** of the Harder-Narasimhan filtration of E. In these terms, a nonzero pure-dimensional sheaf is semistable if and only if its Harder-Narasimhan filtration is of length $\ell(E) = 1$. The ordered $\ell(E)$-tuple

$$\mathrm{HN}(E) = (P(\mathrm{HN}_1(E)), \ldots, P(\mathrm{HN}_\ell(E))) \in \mathrm{HNT}$$

is called the **Harder-Narasimhan type** of E.

In his path-breaking paper [Sh], S.S. Shatz addressed the question of the variation of the Harder-Narasimhan type in a family. The set-up for this is as follows. Let S be a locally noetherian scheme, and let $\pi : X \to S$ be a projective scheme over S, with a relatively ample line bundle $\mathcal{O}_X(1)$. Let E be a coherent sheaf of $\mathcal{O}_X$-modules which is flat over S such each restriction E_s to the schematic fiber $X_s = \pi^{-1}(s)$ is pure-dimensional. The **Harder-Narasimhan function of** E is the function

$$|S| \to \mathrm{HNT} : s \mapsto \mathrm{HN}(E_s)$$

where $|S|$ denotes the underlying topological space of the scheme S. Shatz proved in [Sh] that $\mathrm{HN}(E_s)$ is upper-semicontinuous w.r.t. the partial order $\leq$ on HNT defined above (actually, HN-filtrations in the sense of μ-semistability rather than Gieseker semistability are considered in [Sh], but the proofs in the Gieseker semistability case are similar with obvious changes).

Remark 1 In particular, for any $\tau \in \mathrm{HNT}$, the corresponding level set

$$|S|^{\tau}(E) = \{s \in |S| \text{ such that } \mathrm{HN}(E_s) = \tau\}$$

is locally closed in $|S|$, the subset $|S|^{\leq \tau}(E) = \bigcup_{\alpha \leq \tau} |S|^{\alpha}(E) \subset |S|$ is open in $|S|$, and $|S|^{\tau}(E)$ is closed in $|S|^{\leq \tau}(E)$.

Remark 2 If $(f_1, \ldots, f_p) \in \mathrm{HNT}$, then $(f_2 - f_1, \ldots, f_p - f_1)$ is again in HNT. Let E be pure-dimensional on Y with $\mathrm{HN}(E) \leq (f_1, \ldots, f_p) \in \mathrm{HNT}$. If $E' \subset E$ is a coherent subsheaf with $P(E') = f_1$, then we must have $\mathrm{HN}_1(E) = E'$, that is, such an E' is automatically the maximal destabilizing subsheaf of E. The quotient $E'' = E/E'$ is pure-dimensional, with $\mathrm{HN}(E'') \leq (f_2 - f_1, \ldots, f_p - f_1)$. Moreover, we have $\mathrm{Hom}_Y(E', E'') = 0$.

Remark 3 If $(Y, \mathcal{O}_Y(1))$ is a projective scheme over a field k and if K any extension field of k, then a coherent sheaf E on Y is semistable w.r.t. $\mathcal{O}_Y(1)$ if and only if its base-change $E_K = E \otimes_k K$ to Y_K is semistable w.r.t. $\mathcal{O}_{Y_K}(1) = \mathcal{O}_Y(1) \otimes_k K$. Consequently, if E is any pure-dimensional sheaf on Y then the Harder-Narasimhan filtration $\mathrm{HN}_i(E_K)$ is just the pullback $\mathrm{HN}_i(E) \otimes_k K$ of the Harder-Narasimhan filtration of E.

3. Scheme structures on HN strata

For basic facts that we need from Grothendieck's theory of Quot schemes and their deformation theory, the reader can consult [Hu-Le], [F-G], [Ni 1] and [Ni 2].

Lemma 4 *A morphism $f : T \to S$ between locally noetherian schemes is a closed embedding if (and only if) f is proper, injective, unramified and induces an isomorphism $k(f(t)) \to k(t)$ of residue fields for all $t \in T$.*

Proof Note that $f(T)$ is closed in S, and $f_* \mathcal{O}_T$ is coherent. It only remains to show that the homomorphism $f^{\#} : \mathcal{O}_S \to f_* \mathcal{O}_T$ is surjective. It is enough to show it stalk-wise at all points of $f(T)$, so we can assume that $S = \operatorname{Spec} A$ where A is a

noetherian local ring. Then by finiteness and injectivity of f, we have $T = \operatorname{Spec} B$ where B is a finite local A algebra, and $f^{\#} : A \to B$ is a local homomorphism which by assumption induces an isomorphism $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ on the residue fields, and $\mathfrak{m}_B = \mathfrak{m}_A B$ by assumption of unramifiedness. Hence it follows by the Nakayama lemma that $f^{\#} : A \to B$ is surjective. $\square$

Let X be a projective scheme over a locally noetherian base scheme S, with a chosen relatively ample line bundle $\mathcal{O}_X(1)$. Let E be a coherent sheaf on X which is flat over S, such that the restriction $E_s = E|_{X_s}$ of E to the schematic fiber X_s of X over each $s \in S$ is a nonzero pure-dimensional sheaf of a fixed HN type $\tau = (f_1, \ldots, f_\ell)$. A **relative Harder-Narasimhan filtration** of E is a filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_\ell = E$ by coherent subsheaves on X, such that for each i with $1 \leq i \leq \ell$, the quotient E_i/E_{i-1} is flat S, and for each $s \in S$ this filtration restricts to give the Harder-Narasimhan filtration $\mathrm{HN}_i(E_s)$ of E_s.

We now come to the main result of this note.

Theorem 5 (Main Theorem) *Let X be a projective scheme over a locally noetherian scheme S, with a relatively ample line bundle $\mathcal{O}_X(1)$. Let E be a coherent sheaf on X which is flat over S, such that the restriction E_s is a pure-dimensional sheaf on X_s for each $s \in S$. Let $\tau = (f_1, \ldots, f_\ell) \in \mathrm{HNT}$. Then we have the following.*
(1) Each Harder-Narasimhan stratum $|S|^\tau(E)$ of E has a unique structure of a locally closed subscheme $S^\tau(E)$ of S, with the following universal property: a morphism $T \to S$ factors via $S^\tau(E)$ if and only if the pullback E_T on $X \times_S T$ admits a relative Harder-Narasimhan filtration of type τ.
(2) A relative Harder-Narasimhan filtration on E, if it exists, is unique.
(3) For any morphism $f : T \to S$ of locally noetherian schemes, the schematic stratum $T^\tau(E_T) \subset T$ for E_T equals the schematic inverse image of $S^\tau(E)$ under f.

Proof If $\ell = 1$, then we take $S^\tau(E)$ to be the open subscheme of S consisting of all s such that E_s is semistable with Hilbert polynomial f_1. We now argue by induction on $\ell \geq 2$. By Remark 1, all s with $\mathrm{HN}(E_s) \leq \tau$ form an open subset $|S|^{\leq \tau}(E)$ of S, and $|S|^\tau(E)$ is a closed subset of $|S|^{\leq \tau}(E)$. We give $|S|^{\leq \tau}(E)$ the unique structure of an open subscheme of S, which we denote by $S^{\leq \tau}(E)$. In what follows we will give the closed subset $|S|^\tau(E)$ a particular structure of a closed subscheme of $S^{\leq \tau}(E)$, which has the desired universal property.

Let $X^{\leq \tau}$ be the inverse image of $S^{\leq \tau} = S^{\leq \tau}(E)$ in X, and let $\mathcal{O}_{X^{\leq \tau}}(1)$ and $E^{\leq \tau}$ be the restrictions of $\mathcal{O}_X(1)$ and E to $X^{\leq \tau}$. Consider the relative Quot scheme

$$Q = Quot^{f_\ell - f_1, \mathcal{O}_{X^{\leq \tau}}(1)}_{E^{\leq \tau}/X^{\leq \tau}/S^{\leq \tau}}$$

with projection $\pi : Q \to S^{\leq \tau}$. Then π is projective, hence proper.

Let $q \in Q$ represent a quotient $q' : E_q \to \mathcal{F}$ on X_q. Then $\ker(q') = \mathrm{HN}_1(E_q)$ by Remark 2. If $q \mapsto s \in S^{\leq \tau}$, then by Remark 3 the quotient q' is the pullback of the quotient $E_s \to E_s/\mathrm{HN}_1(E_s)$ which is defined over X_s. Hence the residue field extension $k(s) \to k(q)$ is trivial. By the uniqueness of $\mathrm{HN}_1(E_s)$, there exists at most one such q over s. The fiber of $\pi : Q \to S^{\leq \tau}$ over s is the Quot scheme

$$\pi^{-1}(s) = Quot^{f_\ell - f_1, \mathcal{O}_{X_s}(1)}_{E_s/X_s/k(s)}.$$

By a standard fact in the deformation theory for Quot schemes (see, for example, Theorem 3.11.(2) in [Ni 2]), its tangent space at q is given by

$$T_q(\pi^{-1}(s)) = \mathrm{Hom}_{X_q}(\ker(q'), E_q/\ker(q')) = \mathrm{Hom}_{X_q}(\mathrm{HN}_1(E_q), E_q/\mathrm{HN}_1(E_q))$$

which is zero by Remark 2. Hence $\pi : Q \to S^{\leq \tau}$ is unramified.

It now follows by Lemma 4 that $\pi : Q \to S^{\leq \tau}$ is a closed imbedding.

Now consider the universal quotient sheaf $E_Q \to E''$ on $X_Q = X \times_S Q$. By Remark 2, for all $q \in Q$ the sheaf E''_q on X_q is pure-dimensional, with

$$\mathrm{HN}(E''_q) \leq \tau'' = (f_2 - f_1, \ldots, f_\ell - f_1).$$

In particular, we have $Q^{\leq \tau''}(E'') = Q$. The Harder-Narasimhan type τ'' has length $\ell - 1$, hence by induction on the length, the closed subset $|Q|^{\tau''}(E'')$ of Q has the structure of a closed subscheme $Q^{\tau''}(E'') \subset Q$ which has the desired universal property for E''. We regard Q as a closed subscheme of $S^{\leq \tau}$ via π, and we finally define the closed subscheme $S^\tau(E) \subset S^{\leq \tau}$ by putting

$$S^\tau(E) = Q^{\tau''}(E'') \subset Q \subset S^{\leq \tau}.$$

We now show that $S^\tau(E)$ so defined has the desired universal property. As τ'' has length $\ell - 1$, by induction on the length, the restriction $E''_{S^\tau(E)}$ of E'' to $X \times_S S^\tau(E)$ has a unique relative Harder-Narasimhan filtration

$$0 \subset E''_1 \subset \ldots \subset E''_{\ell-1} = E''_{S^\tau(E)}$$

with Harder-Narasimhan type τ''. For $2 \leq i \leq \ell$, let E_i be the inverse image of E''_{i-1} under the restriction of universal quotient $E_Q \to E''$ to $X_{S^\tau(E)}$. This defines a relative Harder-Narasimhan filtration $0 \subset E_1 \subset \ldots \subset E_\ell = E_{S^\tau(E)}$ of $E_{S^\tau(E)}$ over the base $S^\tau(E)$. In particular, if a morphism $T \to S$ factors via $S^\tau(E)$ then the pullback of this filtration gives a relative Harder-Narasimhan filtration over T.

Conversely, let $f : T \to S$ be a morphism such that the pullback E_T on X_T has a relative Harder-Narasimhan filtration $0 = F_0 \subset F_1 \subset \ldots \subset F_\ell = E_T$ of type

τ. The quotient $E_T \to E_T/F_1$ has Hilbert polynomial $f_\ell - f_1$ over all $t \in T$, so by the universal property of the Quot scheme Q, the morphism $T \to S$ factors via $Q \hookrightarrow S$, inducing a morphism $f' : T \to Q$. By Remark 2, the restriction of E_T/F_1 is pure-dimensional on X_t for all $t \in T$, and $0 = (F_1/F_1) \subset (F_2/F_1) \subset \ldots \subset (F_\ell/F_1) = E_T/F_1$ is a relative Harder-Narasimhan filtration of $E_T/F_1 = (f')^*(E'')$ over the base T, with type τ'' which has length $\ell - 1$. Hence by induction, $f' : T \to Q$ factors via $Q^{\tau''}(E'') = S^\tau(E)$, as desired. This completes the proof of (1).

Next we show the uniqueness of a relative Harder-Narasimhan filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_\ell = E$ over a base S, assuming such a filtration exists. As $|S|^\tau(E) = |S|$, we at least have $S = S^{\le\tau}(E)$. With notation as above, we have shown that $\pi : Q \to S^{\le\tau}$ is a closed imbedding, therefore π admits at most one global section, which shows that E_1 is unique. By inductive assumption on ℓ, the quotient family E/E_1 admits a unique relative Harder-Narasimhan filtration F_j, so defining $E_i \subset E$ to be the inverse image of F_{i-1} under $E \to E/E_1$ for $2 \le i \le \ell$, we see that $0 = E_0 \subset E_1 \subset \ldots \subset E_\ell = E$ is the only possible relative Harder-Narasimhan filtration on E. This proves the statement (2).

The base-change property (3) for the schematic strata is a direct consequence of the universal property (1). This completes the proof of the theorem. $\square$

The following immediate implication of Theorem 5 shows that when the HN type is constant over a reduced base scheme, the outcome is as nice as can be expected.

Corollary 6 (Case of constant HN type over a reduced base) *Let X be a projective scheme over a locally noetherian base scheme S, with a chosen relatively ample line bundle $\mathcal{O}_X(1)$. Let E be a coherent sheaf on X which is flat over S, such that the restriction $E_s = E|_{X_s}$ of E to the schematic fiber X_s of X over each $s \in S$ is a pure-dimensional sheaf of a fixed Harder-Narasimhan type $\tau \in$ HNT. Suppose moreover that S is reduced. Then $S = S^\tau$, that is, E admits a unique relative Harder-Narasimhan filtration.*

4. Moduli stack $Coh^\tau_{X/S}$

For basic terminology and conventions about stacks, we will follow the book [L-MB] by Laumon and Moret-Bailly. In what follows, X will be a projective scheme over a locally noetherian base scheme S, with a chosen relatively ample line bundle $\mathcal{O}_X(1)$, and $\tau = (f_1, \ldots, f_\ell)$ will be any element of HNT.

Let $Coh_{X/S}$ denote the Artin algebraic stack over S of all flat families of coherent sheaves on X/S (see [L-MB] 2.4.4). In any such family, pure-dimensionality of all restriction to fibers is an open condition on the parameter scheme, pure-dimensionality is preserved by arbitrary base changes, and the base-change under a surjection is

pure-dimensional on all fibers if and only if the original is so. Hence pure-dimensional coherent sheaves form an open algebraic substack $Coh^{pure}_{X/S} \subset Coh_{X/S}$.

We will define the moduli stack $Coh^{\tau}_{X/S}$ of pure-dimensional coherent sheaves of type τ as a strictly full sub S-groupoid of $Coh^{pure}_{X/S}$, as follows. For any S-scheme T, we say that an object $E \in Coh^{pure}_{X/S}(T)$ lies in $Coh^{\tau}_{X/S}(T)$ if and only if E admits a relative Harder-Narasimhan filtration with constant type τ. This is clearly closed under pullbacks $f^* : Coh_{X/S}(T) \to Coh_{X/S}(T')$ for all S-morphisms $f : T' \to T$.

To prove that the S-groupoid $Coh^{\tau}_{X/S}$ thus defined is a stack, we need the following property of effective descent.

Lemma 7 *Let T be an S-scheme and let E be an object of $Coh_{X/S}(T)$. Let $f : T' \to T$ be a faithfully flat quasi-compact morphism. If the pullback f^*E is in $Coh^{\tau}_{X/S}(T')$, then E is in $Coh^{\tau}_{X/S}(T)$.*

Proof Each E_t, where $t \in T$, is pure-dimensional with Harder-Narasimhan type τ, as its pullback $E_{t'}$ is so for any $t' \in T'$ with $t' \mapsto t$, and as $T' \to T$ is surjective. It now only remains to construct a relative Harder-Narasimhan filtration of E. This we do by showing that the relative Harder-Narasimhan filtration (F_i) of the pullback $E_{T'}$ descends under $T' \to T$.

Let $T'' = T' \times_T T'$ with projections $\pi_1, \pi_2 : T'' \rightrightarrows T'$. By Grothendieck's result on effective fpqc descent for quasicoherent subsheaves of the pullback of a quasicoherent sheaf, to show that the filtration descends to T we just have to show that the pullbacks of the filtration under the two projections $\pi_1, \pi_2 : T'' \rightrightarrows T'$ are identical. But note that we have an identification $\pi_1^*(E_{T'}) = \pi_2^*(E_{T'}) = E_{T''}$, under which the pullbacks $\pi_1^*(F_i)$ and $\pi_2^*(F_i)$ are relative Harder-Narasimhan filtrations of $E_{T''}$. Hence these filtrations coincide by Theorem 5. $\square$

Theorem 8 *Let X be a projective scheme over a locally noetherian scheme S, with a relatively ample line bundle $\mathcal{O}_X(1)$. Let τ be any Harder-Narasimhan type. Then all flat families of pure-dimensional coherent sheaves on X/S with fixed Harder-Narasimhan type τ form an algebraic stack $Coh^{\tau}_{X/S}$ over S, which is a locally closed substack of the algebraic stack $Coh_{X/S}$ of all flat families of coherent sheaves on X/S.*

Proof The inclusion 1-morphism of S-groupoids $\theta : Coh^{\tau}_{X/S} \hookrightarrow Coh^{pure}_{X/S}$ is fully faithful. Hence $Coh^{\tau}_{X/S}$ is a pre-stack over S. By Lemma 7, the pre-stack $Coh^{\tau}_{X/S}$ satisfies effective fpqc descent, so it is a stack over S. We next prove that it is algebraic.

Given any E in $Coh^{pure}_{X/S}(T)$, let $T^{\tau}(E) \subset T$ be the corresponding schematic Harder-Narasimhan stratum as given by Theorem 5. Let $[E] : T \to Coh^{pure}_{X/S}$ be the

classifying 1-morphism of E. By Theorem 5 we have a natural isomorphism

$$T \times_{[E],Coh^{pure}_{X/S},\theta} Coh^{\tau}_{X/S} \cong T^{\tau}(E)$$

of S-groupoids, under which the projection of the fibered product to T corresponds to the imbedding of $T^{\tau}(E)$ as a locally closed subscheme in T.

This shows the inclusion 1-morphism $\theta : Coh^{\tau}_{X/S} \hookrightarrow Coh_{X/S}$ of stacks is a representable locally closed imbedding. Hence $Coh^{\tau}_{X/S}$ is an algebraic stack over S, which is a locally closed substack of $Coh_{X/S}$. $\square$

We now come to the question of quasi-projectivity of $Coh^{\tau}_{X/S}$. For a given X/S, $\mathcal{O}_X(1)$ and $\tau = (f_1, \ldots, f_\ell)$, consider the following **boundedness condition (*)**.

(*) *There exists a natural number N such that for any morphism* $\operatorname{Spec} K \to S$ *where K is a field and any semistable coherent sheaf F on the base-change X_K whose Hilbert polynomial is equal to f_i for any $1 \leq i \leq \ell$, the sheaf $F(N) = F \otimes \mathcal{O}_{X_K}(N)$ is generated by global sections, and all its cohomology groups $H^j(X_K, F(N))$ vanish for $j \geq 1$.*

By the boundedness theorems of Maruyama-Simpson [Si] and Langer [La], the condition **(*)** is indeed satisfied in many cases of interest, for example, when S is of finite type over an algebraically closed field k of arbitrary characteristic.

Proposition 9 (Quasi-projectivity of $Coh^{\tau}_{X/S}$**)** *If the above boundedness condition* **(*)** *is satisfied, then the stack $Coh^{\tau}_{X/S}$ admits an atlas $U \to Coh^{\tau}_{X/S}$ such that U is a quasi-projective scheme over S.*

Proof If a coherent sheaf E is of type τ on X_K for an S-field K, then by **(*)**, E is a quotient of $\mathcal{O}_{X_K}(-N)^{f_\ell(N)}$. Let Q be the relative Quot scheme over S which parameterizes all coherent quotient sheaves of $\mathcal{O}_X(-N)^{f_\ell(N)}$ on fibers of X/S, with fixed Hilbert polynomial f_ℓ. Let E be the universal quotient sheaf on $X \times_S Q$. Let $Q_o \subset Q$ be the open subscheme consisting of all $q \in Q$ satisfying the conditions that E_q is pure-dimensional, $E_q(N)$ is generated by global sections, the map $H^0(X_q, \mathcal{O}^{f_\ell(N)}_{X_q}) \to H^0(X_q, E_q(N))$ induced by q is an isomorphism, and $H^i(X_q, E_q(N)) = 0$ for all $i \geq 1$ (each of these conditions is an open condition).

Let E_o be the restriction of E to $X \times_S Q_o$. Let $Q^{\tau}_o(E_o)$ be the locally closed subscheme of Q_o corresponding to the Harder-Narasimhan type τ, given by Theorem 5. The classifying 1-morphism $[E_o] : Q^{\tau}_o(E_o) \to Coh^{\tau}_{X/S}$ of E_o is an atlas for $Coh^{\tau}_{X/S}$ (that is, $[E_o]$ is a representable smooth surjection), as follows from the proof of Theorem 4.6.2.1 in Laumon and Moret-Bailly [L-MB]. As Q is projective over S, its locally closed subscheme U is quasi-projective over S, as desired. $\square$

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School of Mathematics,
Tata Institute of Fundamental Research,
Homi Bhabha Road,
Mumbai 400 005,
INDIA.

nitsure@math.tifr.res.in

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